TRAVERSE COMPUTATION:
ELLIPSOID VERSUS THE UTM PROJECTION

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ABSTRACT

Given a set of traverse measurements reduced to the ellipsoid, grid coordinates (E, N) can be computed in two ways: (i) reduce the traverse measurements to the Universal Transverse Mercator (UTM) projection plane and then use plane trigonometry or (ii) compute geodetic coordinates (ϕ, λ) directly using the *direct* and *inverse* cases on the ellipsoid and then transform these to grid coordinates. The first method (traverse computation on the UTM plane) requires iteration and is slow; the second method (traverse computation on the ellipsoid) is simpler and quicker.

The Intergovernmental Committee on Surveying and Mapping (ICSM) and Geoscience Australia have provided Microsoft® Excel spreadsheets for the calculations and several authors have presented information on the two methods using these software tools. This paper briefly describes these methods with some technical and historical information regarding geodesic curves and the direct and inverse cases on the ellipsoid.

INTRODUCTION

In Australia, topographic mapping and coordination is based on rectangular coordinate grids (east *E,* north *N*) overlaying conformal projections of latitudes ϕ and longitudes λ of points related to geodetic datums. There are <u>two</u> geodetic datums of interest: the new Geocentric Datum of Australia (GDA) and the old Australian Geodetic Datum (AGD), <u>one</u> conformal map projection: the UTM, and <u>two</u> grids: the new Map Grid Australia (MGA) and the old Australian Map Grid (AMG). Hence we have the coordinate "pairs" AGD/AMG and GDA/MGA. There have been several "realizations" of geodetic datums in Australia – a realization being the actual determination of coordinates (ϕ, λ) related to a reference ellipsoid, by the mathematical adjustment of measurements between stations in the national geodetic network. The first of these was in 1966 and the second in 1984; both being realizations of the AGD and known as AGD66 and AGD84 with grid coordinates designated AMG66 and AMG84. The AGD is a topocentric datum that has now been superseded by the GDA with a realization designated GDA94 with grid coordinates MGA94. [In 1995 the Australian government proclaimed the new datum and produced a geodetic coordinate set designated GDA94 referred to the reference ellipsoid of the Geodetic Reference System 1980 (GRS80) and located with respect to the International Terrestrial Reference Frame 1992 (ITRF92) at the epoch 1994.0.]

In Australia, coordinate transformations $(\phi, \lambda \Leftrightarrow E, N)$ as well as calculation of grid convergence γ and point scale factor k are defined by Redfearn's formula (Redfearn 1948). Calculations using these formula can be easily done using Microsoft® Excel spreadsheets available on-line via the Internet at the Geoscience Australia website (http://www.ga.gov.au/) following the links to Geodetic Calculations then Calculate Bearing Distance from Latitude Longitude. At this web page the spreadsheet Redfearn.xls is available for use or downloading. Alternatively, the ICSM has produced an on-line publication *Geocentric Datum of Australia Technical Manual* Version 2.2 (GDA Technical Manual, ICSM 2002) with a link to Redfearn.xls

Computations on the reference ellipsoid are divided into two cases, (i) the *direct* case: given ϕ, λ of point 1 and the azimuth α and geodesic distance s to point 2, compute ϕ, λ of point 2, and (ii) the *inverse* case: given ϕ, λ of points 1 and 2, compute the azimuth and geodesic distance between them. The direct and inverse cases on the ellipsoid are equivalent to the familiar plane coordinate calculations "radiations" and "joins". Excel spreadsheets for the direct and inverse cases on the ellipsoid are available at the Geoscience Australia website following the links to Geodetic Calculations then Calculate Bearing Distance from Latitude Longitude. At this web page the spreadsheet Vincenty.xls is available for use or downloading. Alternatively, the GDA Technical Manual has a link to Vincenty.xls

The GDA Technical Manual is a source of valuable information, references and computation formula, guidelines and Excel spreadsheets. Also, two recent publications are very useful; one in the *Trans Tasman Surveyor* by Will Featherstone and Jean Rüeger (Featherstone & Rüger 2000) and the other in *The Australian Surveyor* by Will Featherstone and John Kirby (Featherstone & Kirby 2002). These papers describe the reduction of traverse measurements to the ellipsoid and traverse computations on the ellipsoid and UTM plane. [It should be noted, that in the latter of these two papers there is an error in the description of the process required for computation on the map plane.]

In addition to these publications, the present author has provided a document: *Traverse Computation on the Ellipsoid and on the Universal Transverse Mercator projection* (Deakin 2005) for distribution at this conference; and will gladly receive comment on its usefulness (or otherwise) to the profession.

Two of the aforementioned publications, Featherstone & Kirby 2002 and Deakin 2005, make it clear that traverse computation on the ellipsoid is a quicker and more direct method than traverse computation on the UTM plane. Indeed both publications use the same traverse to demonstrate the methods; the latter has 16 pages devoted to a detailed computation on the UTM plane and only 3 pages for a computation on the ellipsoid and the former publication states that the time to compute on the plane was approximately 60 minutes versus 20 minutes for the ellipsoid. Remarkable savings in time and effort.

The following two sections have a brief outline of the two methods of computation.

TRAVERSE COMPUTATION ON THE UTM PLANE

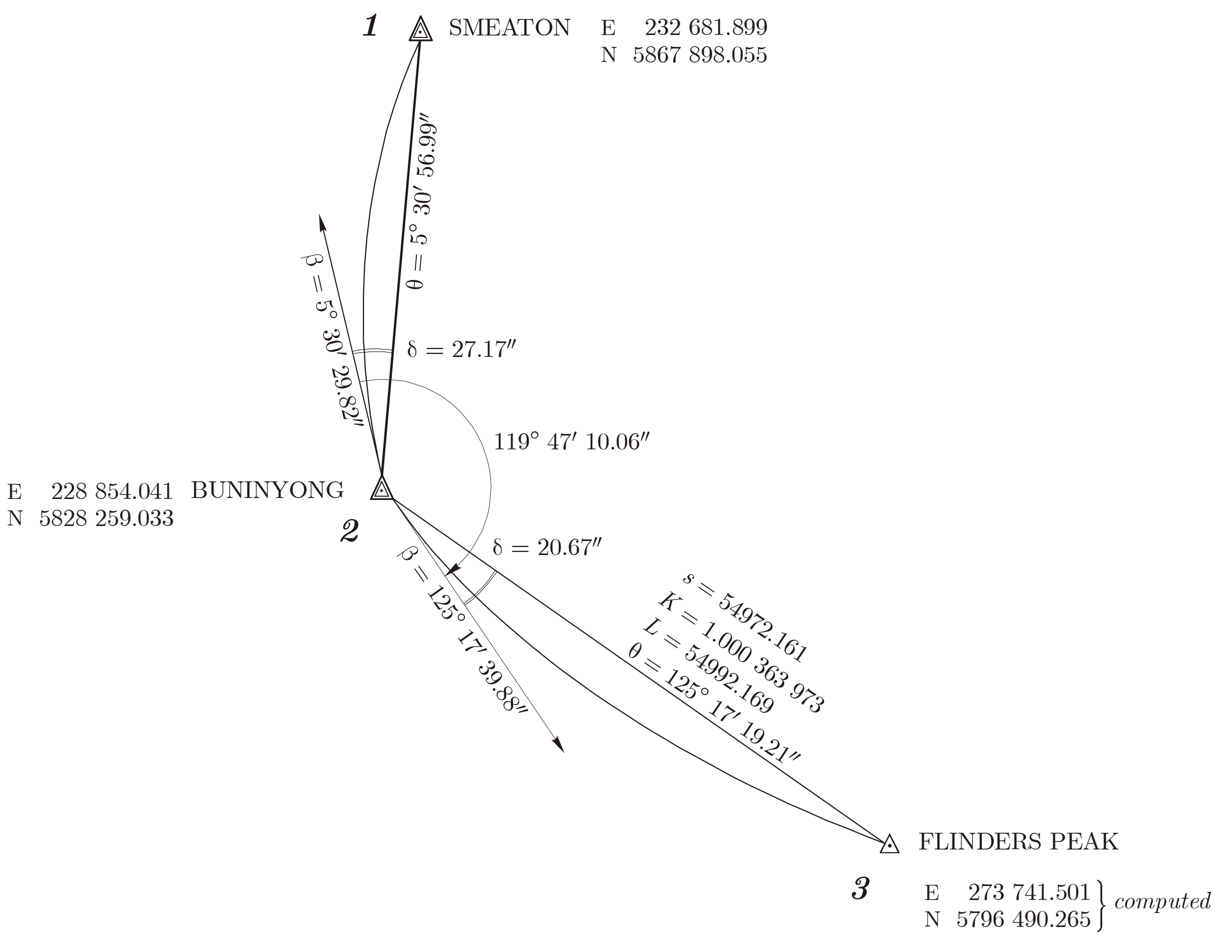

Figure 1. Traverse *Smeaton(1)–Buninyong(2)–Flinders Peak(3)*

Figure 1 shows a traverse *Smeaton(1)–Buninyong(2)–Flinders Peak(3). Smeaton* and *Buninyong* are fixed stations (known coordinates) and *Flinders Peak* is a floating station (unknown coordinates). It is required to calculate the grid coordinates of *Flinders Peak.*

The coordinates shown in Figure 1 are MGA94 Zone 55 and the central meridian of the zone is to the east (right-hand side of the page). The quasi-observations are the ellipsoidal angle at *Buninyong(2):* $\psi_2 = 119^\circ\, 47'\, 10.06''$ and the geodesic distance *Buninyong(2)–Flinders Peak(3):* $s_{23} = 54972.161\,\mathrm{m}$, and it is assumed that these are the result after all proper corrections have been applied to the actual field measurements. The coordinates of *Flinders Peak(3)* are computed in the following steps: [detailed results are shown in Deakin (2005)]

(1) Compute the *plane bearing* θ_{21} of the back-sight *Buninyong(2)–Smeaton(1).*
This is a simple plane coordinate calculation using the known MGA94 coordinates.
$\theta_{21} = 5^\circ\, 30'\, 56.99''$

(2) Compute the *arc-to-chord* correction δ_{21} of the back-sight *Buninyong(2)–Smeaton(1).*
The calculation of δ_{21} requires the mean radius r_m , a function of the mean latitude ϕ_m of the back-sight line. ϕ_m can be calculated using Redfearn.xls to transform $E, N \Rightarrow \phi, \lambda$ for both ends of the line, taking the mean and then computing r_m followed by $\delta_{21} = 27.17''$.

(3) Compute the *grid bearing* β_{21} of the back-sight *Buninyong(2)–Smeaton(1).*
$\beta_{21} = \theta_{21} - \delta_{21} = 5^\circ\, 30'\, 29.82''$

(4) Compute the *grid bearing* β_{23} of the forward-sight *Buninyong(2)–Flinders Peak(3).*
$\beta_{23} = \beta_{21} + \psi_2 = 125^\circ\, 17'\, 39.88''$ [Note that β_{23} is now known exactly]

(5) Compute the *E,N* coordinates of *Flinders Peak(3)*
This step requires iteration.

(5.1) The *E,N* coordinates of *Flinders Peak* are computed using plane trigonometry and the *plane bearing* θ_{23} and *plane distance* L_{23} are required. They are unknown, but can be approximated by $\theta_{23} \simeq \beta_{23}$ and $L_{23} \simeq k_2\, s_{23}$ where k_2 is the *point scale factor* at *Buninyong,* a quantity that can be calculated by using Redfearn.xls to transform $E, N \Rightarrow \phi, \lambda$ and k, γ for *Buninyong* (γ is the grid convergence). Using these approximations and plane trigonometry, approximate coordinates of *Flinders Peak* can be calculated.

(5.2) Use Redfearn.xls to transform $E,N \Rightarrow \phi,\lambda$ for *Flinders Peak,* then compute ϕ_m and r_m for the line *Buninyong(2)–Flinders Peak(3);* then compute the *arc-to-chord correction* δ_{23} and *line scale factor* K_{23}. These will be approximate values (since *E,N* of *Flinders Peak* and r_m of the line are approximations) but will give better estimates of $\theta_{23} = \beta_{23} + \delta_{23}$ and $L_{23} = K_{23}\, s_{23}$.

(5.3) Compute "improved" coordinates of *Flinders Peak* using plane trigonometry and θ_{23}, L_{23} from step (5.2).

Repeat steps (5.2) and (5.3) until there is no change in the coordinates of *Flinders Peak.*

TRAVERSE COMPUTATION ON THE ELLIPSOID

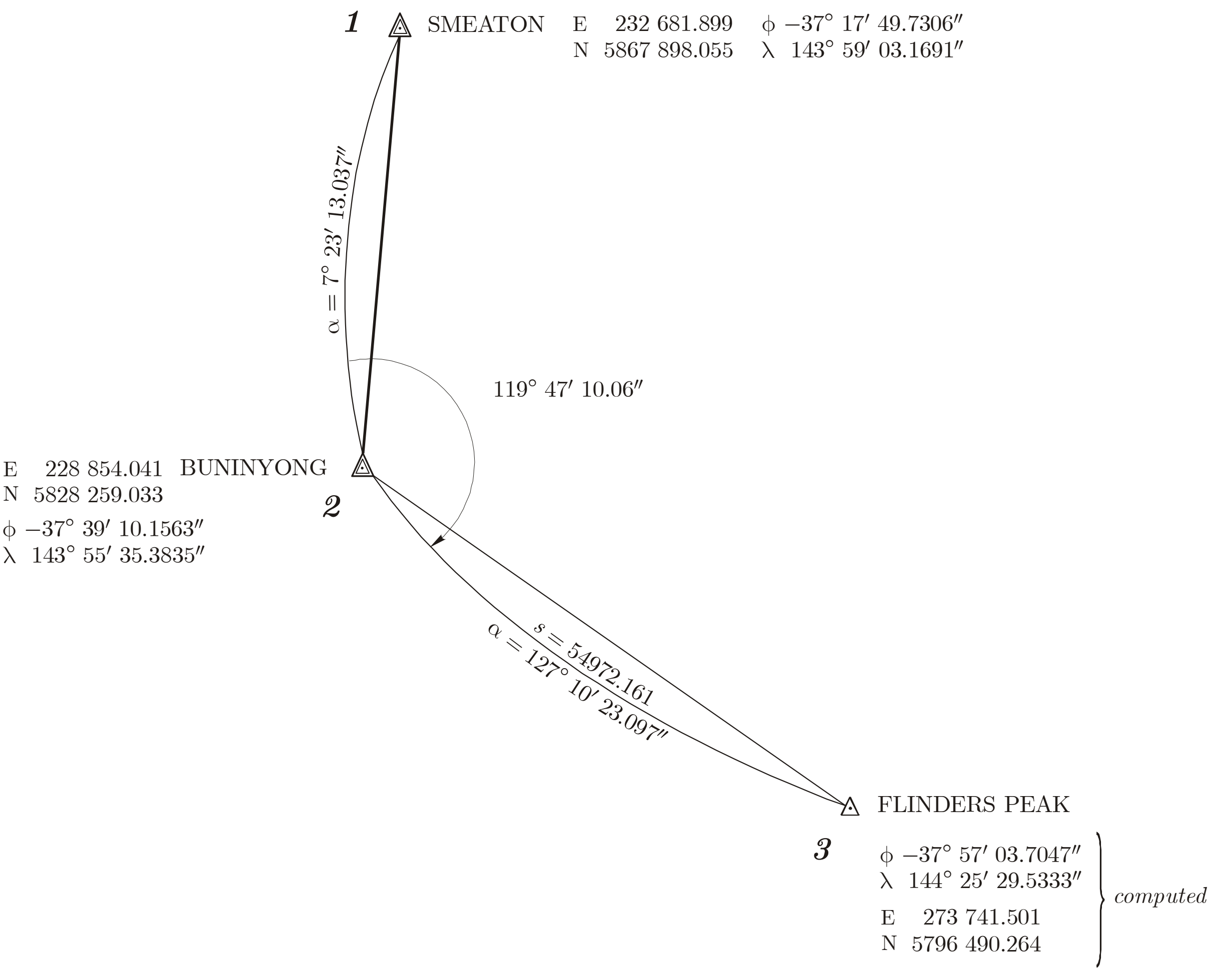

Figure 2. Traverse *Smeaton(1)–Buninyong(2)–Flinders Peak(3)*

Figure 2 shows the same traverse as before; *Smeaton(1)–Buninyong(2)–Flinders Peak(3). Smeaton* and *Buninyong* are fixed stations (known coordinates) and *Flinders Peak* is a floating station (unknown coordinates). It is required to calculate the grid coordinates of *Flinders Peak.*

As before, the *E,N* coordinates shown in Figure 2 are MGA94 Zone 55 and the central meridian of the zone is to the east (right-hand side of the page). The quasi-observations are the ellipsoidal angle at *Buninyong(2):* $\psi_2 = 119^\circ\, 47'\, 10.06''$ and the geodesic distance *Buninyong(2)–Flinders Peak(3):* $s_{23} = 54972.161\,\mathrm{m}$, and it is assumed that these are the result after all proper corrections have been applied to the actual field measurements. The coordinates of *Flinders Peak* are computed in the following steps: [detailed results are shown in Deakin (2005)]

(1) Transform $E, N \Rightarrow \phi, \lambda$ for *Buninyong* and *Smeaton.*
Use Redfearn.xls

(2) Compute the azimuth α_{21} of the back-sight *Buninyong–Smeaton.*
Use Vincenty.xls (Inverse Case): $\alpha_{21} = 7^\circ\, 23'\, 13.037''$

(3) Compute the azimuth α_{23} of the forward-sight *Buninyong–Flinders Peak.*
$\alpha_{23} = \alpha_{21} + \psi_2 = 127^\circ\, 10'\, 23.097''$

(4) Compute ϕ, λ for *Flinders Peak.*
Use Vincenty.xls (Direct Case) with $\alpha_{23} = 127^\circ\, 10'\, 23.097''$ and $s_{23} = 54972.161\,\mathrm{m}$

(5) Transform $\phi, \lambda \Rightarrow E, N$ for *Flinders Peak.*
Use Redfearn.xls

This is a much simpler method than computing on the UTM plane.

The simplicity of this method is due in no small part to the availability of Excel solutions of the *direct* and *inverse* cases on the ellipsoid. These two cases are fundamental geodetic operations and may be thought of as ellipsoidal equivalents of the plane coordinate operations radiations and joins. The direct and inverse cases are based on the properties of a geodesic on an ellipsoid and the following section contains an outline of the fundamental properties of this particular curve. Parts of this section have been taken from a publication, currently in press, by this author and Dr Max Hunter of the School of Mathematical and Geospatial Sciences, RMIT University.

THE GEODESIC ON THE ELLIPSOID

The *geodesic* is a unique curve on the surface of an ellipsoid defining the shortest distance between two points. A geodesic will cut meridians of an ellipsoid at angles α , known as *azimuths* and measured clockwise from north 0° to 360°. Figure 3 shows a geodesic curve C between two points A (ϕ_A, λ_A) and B (ϕ_B, λ_B) on an ellipsoid. The geodesic curve C, of length s, from A to B has a *forward azimuth* α_{AB} measured at A and a *reverse azimuth* α_{BA} measured at B.

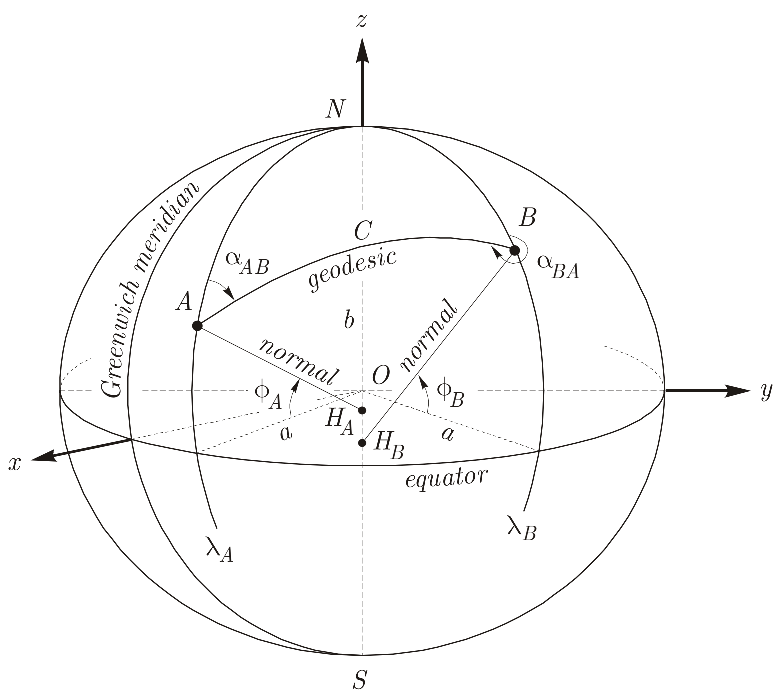

Figure 3. Geodesic curve C on an ellipsoid

It is interesting to note that the geodesic on a plane is a straight line and on a sphere, the geodesic is a great circle. On these two simple surfaces, plane and spherical trigonometry respectively are used to compute direction and distance. The ellipsoid is a slightly more complicated surface (a surface of revolution created by rotating an ellipse about its minor axis) and the geodesic is a curve having curvature in two directions and having a *characteristic equation*

$$w \sin\alpha = \nu \cos\phi \sin\alpha = C \tag{1}$$

where ν is the radius of curvature of the ellipsoid in the prime vertical plane and $w = \nu \cos\phi$ is the radius of the parallel of latitude. Equation (1) is known as *Clairaut's equation* in honour of the French mathematical physicist Alexis-Claude Clairaut (1713-1765). In a paper in 1733 titled *Détermination géométrique de la perpendiculaire à la méridienne, tracée par M. Cassini, avec plusieurs methods d'en tirer la grandeur et la figure de la terre* (Geometric determination of the perpendicular to the meridian, traced by Mr. Cassini, … on the figure of the Earth.) Clairaut made an elegant study of the geodesics of quadrics of rotation (DSB 1971).

The characteristic equation of a geodesic shows that the geodesic on the ellipsoid has the intrinsic property that at any point, the product of the radius w of the parallel of latitude and the sine of the azimuth of the geodesic at that point is a constant. This means that as $w = \nu \cos\phi$ decreases in higher latitudes, in both the northern and southern hemispheres, $\sin\alpha$ increases until it reaches a maximum or minimum of ± 1, noting that the azimuth of a geodesic at a point will vary between $0°$ and $180°$ if the point is moving along a geodesic in an easterly direction or between $180°$ and $360°$ if the point is moving along a geodesic in a westerly direction. At the point when $\sin\alpha = \pm 1$, which is known as the *vertex*, w is a minimum and the latitude ϕ will be a maximum value ϕ_0, known as the geodetic latitude of the vertex. Thus, the geodesic oscillates over the surface of the ellipsoid between two parallels of latitude having a maximum in the northern and southern hemispheres and crossing the equator at nodes; but due to the eccentricity of the ellipsoid the geodesic will not repeat after a complete revolution.

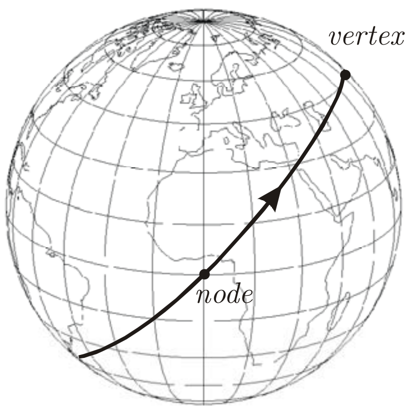

Figure 4a

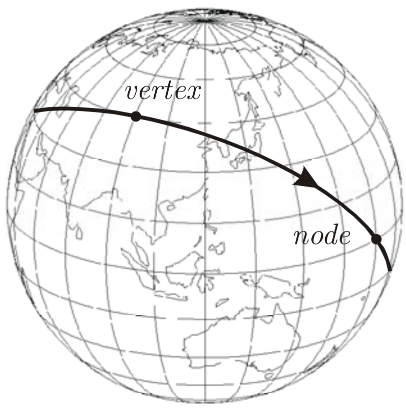

Figure 4b

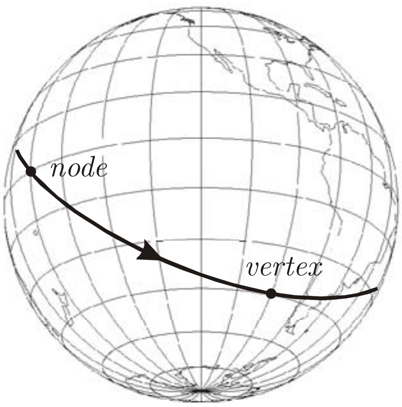

Figure 4c

Figures 4a, 4b and 4c show a single revolution of a geodesic on the Earth. The geodesic reaches maximum latitudes of approximately $\pm 45°$ and has an azimuth of approximately $45°$ as it crosses the equator at longitude $0°$.

Figure 5 shows a schematic representation of the oscillation of a geodesic on an ellipsoid. P is a point on a geodesic that crosses the equator at A, heading in a north-easterly direction reaching a maximum northerly latitude ϕ_{max} at the vertex P_0 (north), then descends in a south-easterly direction crossing the equator at B, reaching a maximum southerly latitude ϕ_{min} at P_0 (south), then ascends in a north-easterly direction crossing the equator again at A'. This is one complete revolution of the geodesic, but $\lambda_{A'}$ does not equal λ_A due to the eccentricity of the ellipsoid, hence we say that the geodesic curve does not repeat after a complete revolution.

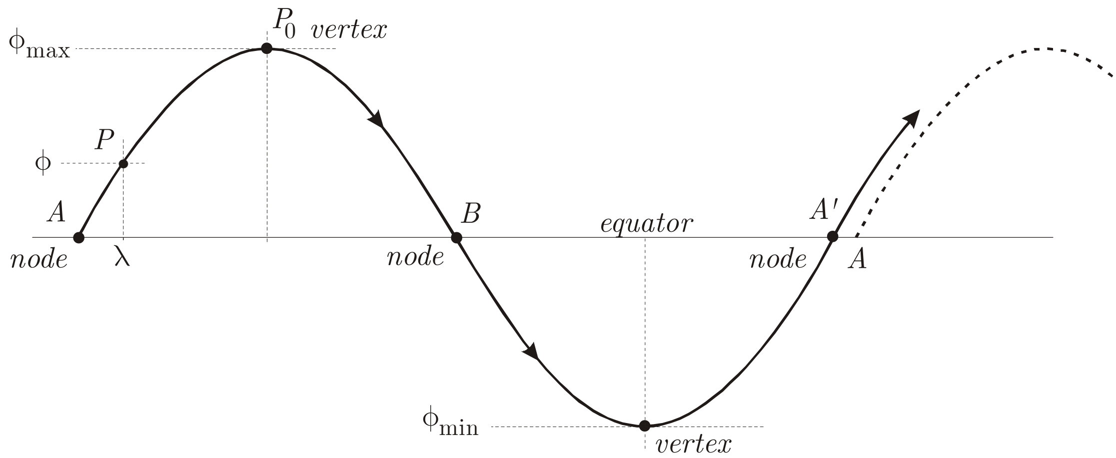

Figure 5. Schematic representation of the oscillation of a geodesic on an ellipsoid

It is interesting to note that the geodesic passing through P at latitude 9° 35' 24" North and having an azimuth at P of 43° 12' 36" on the GRS80 ellipsoid will wrap around the ellipsoid reaching vertices at $\phi_0 = \pm 47^\circ\, 37'\, 42.820248''$ and will have a longitude difference at the nodes A and A' of $\Delta\lambda = \lambda_A - \lambda_{A'} = 0^\circ\, 48'\, 52.431555''$. This equates to a difference of 90676.885 metres along the equator.

THE DIRECT AND INVERSE PROBLEMS ON THE ELLIPSOID: A Limited History

The *direct* problem on an ellipsoid is: given latitude and longitude of A and the azimuth α_{AB} and geodesic distance s_{AB}, compute the latitude and longitude of B and the reverse azimuth α_{BA}. The *inverse* problem is: given the latitudes and longitudes of A and B, compute the forward and reverse azimuths α_{AB}, α_{BA} and the geodesic distance s_{AB}.

The direct and inverse problems of the geodesic on an ellipsoid are fundamental geodetic operations that have been studied in detail by geodesists and surveyors since the early 1800's.

The nature of an ellipsoid and of the geodesic upon it, does not allow direct mathematical solutions to the problems; instead, they are solved by series expansions of integrals that are the solutions of differential equations. The first of these was due to Friedrich Wilhelm Bessel, the German astronomer, mathematician and surveyor who published a method of solution in *Astronomische Nachrichten* (Astronomical Notes, Vol. 4, p.241) in 1823. Bessel's method, using differential relationships between geodesic elements and the corresponding elements on an auxiliary spherical triangle, is not limited by distance provided a suitable number of terms are included. Similar methods were published in the late 1800's by the Clarke (1880) and Helmert (1880); Krakiwisky and Thompson (1974) have a detailed development of Bessel's method. In 1934, G.T. McCaw published an improved method of solving the direct case (McCaw 1934) and H.F. Rainsford (1955) published a detailed treatment of long geodesics on the ellipsoid, using McCaw's solution for the direct case and revising previously published methods for the inverse case. Rainsford notes in his paper that the methods of solution involve a tedious process, particularly with 10-place logarithms.

Prior to the advent of computers, practical solutions were devised for certain lengths of geodesic lines, and as a rule: the shorter the line, the easier the formulae (and workload). But, using short-line formulae on a long-line problem often proved disastrous, since approximations appropriate for a short line on an ellipsoid are rarely appropriate for a long line and there was no "one solution fits all". In the latter part of the 20th century with the burgeoning availability of powerful desktop computers, solutions for the direct and inverse cases using series formula based on the solution of differential equations became practicable. One of these, a solution developed by the American geodesist Thaddeus Vincenty (Vincenty 1975) and based on Rainsford's work, has been designed for ease of programming and may be used for lines ranging from a few cm to nearly 20,000 km with mm precision. The ICSM has programmed Vincenty's method to run on Microsoft Excel spreadsheets and has made available a workbook, Vincenty.xls, containing spreadsheets for the direct and inverse cases (ICSM 2002). These spreadsheets are easy to use, and combined with $\phi, \lambda \Leftrightarrow E, N$ transformations using Redfearn.xls, geodetic traverse computations are now relatively routine operations.

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